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MATHEMATICAL ANALYSIS OF THE COBWEB MODEL

Hubal H.M. Mathematical analysis of the cobweb model. The paper explores the cobweb model. Mathematical analysis of this model is conducted.
Keywords: cobweb model, equilibrium price, demand and supply functions, recurrence relation.

Губаль Г. М. Математичний аналіз павутинної моделі. У статті досліджується павутинна модель. Здійснено математичний аналіз даної моделі.
Ключові слова: павутинна модель, рівноважна ціна, функції попиту та пропозиції, рекурентне співвідношення.

Губаль Г. Н. Математический анализ паутинной модели. В статье исследуется паутинная модель. Проведено математический анализ данной модели.
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Introduction. Every producer tries to purchase his goods as expensively as possible, and every customer tries to buy goods as cheaply as possible.

The establishing of the equilibrium price is one of the main tasks of markets. From an economic point of view, the equilibrium state is a necessary condition for stable and normal functioning of economics. Competitive equilibrium cannot be achieved automatically and through the uncoordinated actions of participants in the economic processes of production and consumption in their own interest. Is there a possibility of achieving the equilibrium state in real economic systems? From the point of view of the modern dynamical systems theory, it is necessary for establishing the equilibrium price that the equilibrium state be globally stable.

The main part. Consider two main categories of market relations, namely the demand and supply, depending on many factors. A chief factor is a commodity price.

Denote the commodity price by *p*, the demand volume by *d*, the supply volume by *s*, the commodity volume by *q*.

For small *p*, we have:

$$d(p)-s(p)=f(p)>0 \text{ (the demand exceeding the supply),}$$

for large *p*, we have:

$$d(p)-s(p)=f(p)<0 \text{ (the supply exceeding the demand).}$$

Considering $d(p)$ and $s(p)$ to be continuous functions, we conclude that the function $f(p)$ is also continuous (by the theorem concerning arithmetic operations on continuous functions). Changing sign, i.e. from $f(p)>0$ to $f(p)<0$, the function $f(p)$ passes through a zero value (there exists a root), i.e.:

$$d(p)-s(p)=f(p)=0 \quad \text{or} \quad d(p)=s(p).$$

Let p^* be the root of this equation. Therefore, there exists the price p^* such that $d(p^*)=s(p^*)$, i.e. the demand equals the supply. The price p^* is equilibrium; the demand and supply are equilibrium at this price.

Consider a model of search equilibrium price the so-called cobweb model (the graphic illustration of the process of searching equilibrium price resembles a cobweb) [1-4]. It explains the cycles of charge in sales volumes and prices regularly repeating themselves.

Assume that a decision on the commodity volume is made depending on the last period's commodity price. So, the cultivated area intended for the crop can be selected depending on crop prices prevailed in the previous year.

Consider the situation shown in fig. 1.

Let the commodity supply volume be $s(p_0)=s_1=q_1$ at the initial point depending on the last period's commodity price p_0. The price p_0 being greater than the equilibrium price, the demand price p_1, being lower than the equilibrium price, is paired with the volume q_1 on the demand curve $d(p)$. The price p_1 is paired with the supply volume $s(p_1)=s_2=q_2$ on the supply curve $s(p)$, leading to increase of the demand price to the magnitude p_2. The price p_2 is paired with the supply volume $s(p_2)=s_3=q_3$ on the supply curve $s(p)$ etc. In this case, 'the spiral' converges to the equilibrium point $(p^*; q^*)$.

Thus, in time period t the producer (an agricultural enterprise) determines the commodity volume s_t based on the price set in last time period $(t-1)$, i. e. $s_t = s(p_{t-1})$. We can take, for example, one year per time unit, namely the production cycle has a duration of one year (as a rule, agricultural enterprises), and the decision on the market's supply volume is made at the end of this cycle. The commodity demand volume depending on this period's commodity price, $d_t = d(p_t)$. Therefore, trade dynamics in the cobweb model can be described by the system of equations:

$$\begin{cases} s_t = s(p_{t-1}); \\ d_t = d(p_t); \\ d_t = s_t = q_t \end{cases}$$

or by one equation:

$$d(p_t) = s(p_{t-1}), \qquad (1)$$

where t are discrete time moments at which price changes occur $(t = 1, 2, 3, \ldots)$.

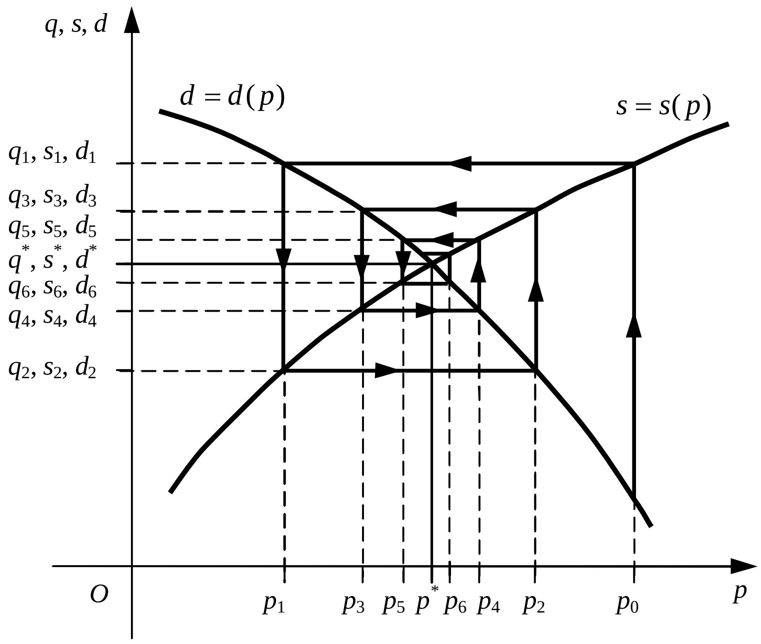

Fig. 1

Using this equation, we can find the value of the price p_t in time period t by means of the known value of the price p_{t-1} in time period $(t-1)$. Expressing p_t in terms of p_{t-1}, by formula (1), we obtain a recurrence relation.

The process of establishing the equilibrium price for this model in accordance with (1) is as follows.

The commodity price p_0 is set at the initial moment of time. If $d(p_0) > s(p_0)$ (the demand exceeding the supply), then the price rises to p_1, the demand therefore decreasing to the supply. If $d(p_0) < s(p_0)$ (oversupply (fig. 1)), then the price falls to p_1, the demand therefore increasing to the supply. The equality $d(p_1) = s(p_0)$ must be fulfilled. Similarly, the following cycles of the process for establishing the equilibrium price are fulfilled.

In accordance with formula (1) obtained (fig. 1), we have:

$$d(p_1) = s(p_0) = q_1, \quad d(p_2) = s(p_1) = q_2,$$

$$d(p_3) = s(p_2) = q_3, \quad d(p_4) = s(p_3) = q_4 \text{ etc.}$$

Evidently, $p^* = \lim_{t\to\infty} p_t$, where $t = 1, 2, 3, \ldots$.

In this case, ‘the cobweb’ is called a convergent or dynamic cobweb.

Besides, the described ‘spiral’ does not always ‘curl’. In some cases it can ‘uncurl’, i.e. it can be divergent, as shown in fig. 2.

What properties of functions $d(p)$ and $s(p)$ does the convergence or divergence of ‘the spiral’ described above depend on? It is rather a difficult question. Confine ourselves to naming one of the factors influencing the convergence. It is the so-called elasticity of demand and supply.

Another case concerns the situation when the angles of inclination of the rectilinear functions of supply and demand are related by the expression $\beta_d = \pi - \alpha_s$ (fig. 3). Thus, ‘the cobweb’ neither converges nor diverges but forms a single closed cycle on which each subsequent cycle is superimposed (perfect ‘cobweb’), namely the market prices will regularly fluctuate with the constant amplitude $\mathbb{A} = p_0 - p^*$.

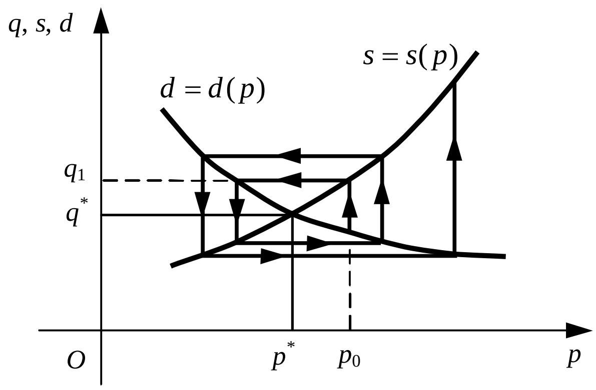

Fig. 2

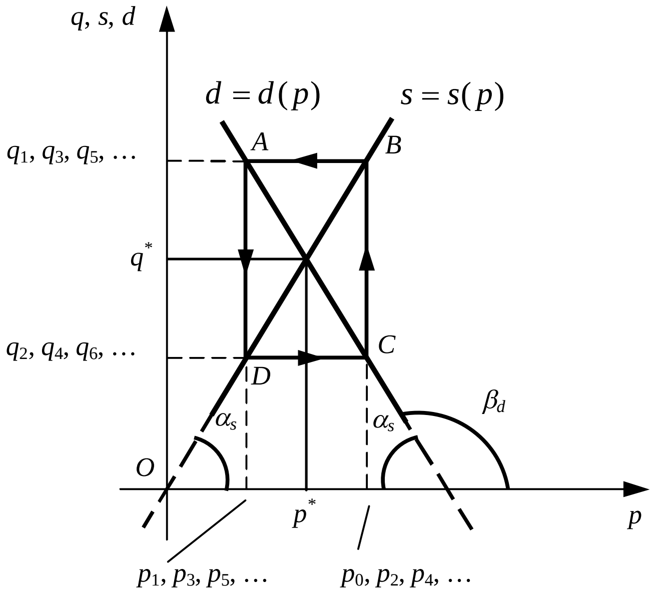

Fig. 3

If in fig. 1, for instance, the points $(p_1; q_1), (p_3; q_3), (p_5; q_5), \ldots$ are different, then in fig. 3 the points $(p_1; q_1), (p_3; q_3), (p_5; q_5), \ldots$ are identical and represent the point *A*. This is analogous to the points *B*, *C* and *D*.

Consider, for example, the case of the cobweb model with the linear functions of demand and supply**:**

$$s(p)=s(p_{t-1})=a+bp_{t-1}, \quad d(p)=d(p_t)=c-lp_t.$$

Here $b>0$, as the supply function increases; $l>0$, as the demand function decreases; taking into account that $s(0)\geq 0$ and assuming that the demand exceeds the supply at a zero price, we write: $d(0)>s(0)\geq 0$, i.e. $c>a\geq 0$.

We write the condition of equilibrium (see formula (1)):

$$d(p_t)=s(p_{t-1})$$

or

$$c-lp_t=a+bp_{t-1}. \qquad (2)$$

First, we find the equilibrium price p^* and the equilibrium demand volume, the equilibrium supply volume. They satisfy the equation $d^*=s^*$ or the equation $c-lp^*=a+bp^*$, whence

$$p^*=\frac{c-a}{b+l}, \quad d^*=s^*=a+bp^*=\frac{al+bc}{b+l}.$$

We investigate the trend of prices and volumes of supply and demand in the case when the initial point does not coincide with the equilibrium one. The problem can be solved graphically. This gives 'the cobweb' confirming the name of the model.

Setting, on the supply straight line, the initial point $(p_0;q_1)$, not coinciding with the equilibrium point $(p^*;q^*)$, we put consistently points in accordance with the calculation procedure on the cobweb model and connect them by horizontal and vertical straight lines. From the graphical analysis, we can obtain the following result.

If the demand straight line is steeper than the supply straight line ($b<l$), then the equilibrium will be stable in the market (fig. 4).

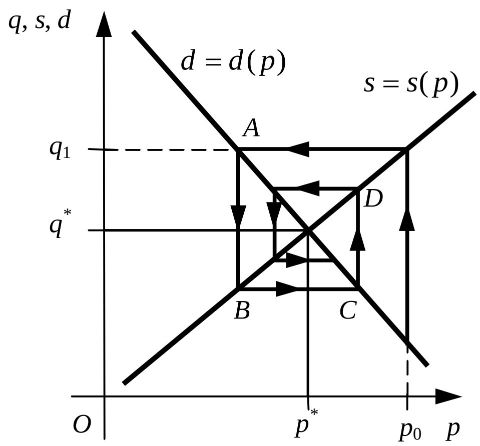

Fig. 4

If the supply straight line is steeper than the demand straight line ($b>l$), then the equilibrium can be unstable in the market (fig. 5).

If the demand and supply straight lines have exactly the same slope ($b=l$), then the market prices will regularly fluctuate with the constant amplitude $\mathbb{A}=p_0-p^*$ (the indifferent equilibrium, fig. 3).

We note that the slope of the straight line $d(p)$ is steeper than the slope of the straight line $s(p)$ (fig. 4). This follows from the fact that the cathetus *AB* is greater than the cathetus *CD* in the right-angled triangles *ABC* and *BCD* and, therefore, the hypotenuse *AC* is inclined steeper than the hypotenuse *BD*.

We conduct the mathematical analysis of this model. From formula (2), we obtain the recurrence relation:

$$p_t = \frac{c-a}{l} - \frac{b}{l} p_{t-1}.$$

Applying consistently the recurrence relation, we obtain:

$$p_1 = \frac{c-a}{l} - \frac{b}{l} p_0,$$

$$p_2 = \frac{c-a}{l} - \frac{b}{l}\left[\frac{c-a}{l} - \frac{b}{l} p_0\right] = \frac{c-a}{l}\left[1-\frac{b}{l}\right] + \left(\frac{b}{l}\right)^2 p_0,$$

$$p_3 = \frac{c-a}{l} - \frac{b}{l}\left[\frac{c-a}{l}\left[1-\frac{b}{l}\right] + \left(\frac{b}{l}\right)^2 p_0\right] = \frac{c-a}{l}\left[1-\frac{b}{l}+\left(\frac{b}{l}\right)^2\right] - \left(\frac{b}{l}\right)^3 p_0,$$

$$p_4 = \frac{c-a}{l} - \frac{b}{l}\left[\frac{c-a}{l}\left[1-\frac{b}{l}+\left(\frac{b}{l}\right)^2\right] - \left(\frac{b}{l}\right)^3 p_0\right] =$$

$$= \frac{c-a}{l}\left[1-\frac{b}{l}+\left(\frac{b}{l}\right)^2 - \left(\frac{b}{l}\right)^3\right] + \left(\frac{b}{l}\right)^4 p_0$$

or in the general case:

$$p_t = \frac{c-a}{l}\left[1-\frac{b}{l}+\left(\frac{b}{l}\right)^2 - \left(\frac{b}{l}\right)^3 + \cdots + (-1)^{t-1}\left(\frac{b}{l}\right)^{t-1}\right] + (-1)^t\left(\frac{b}{l}\right)^t p_0.$$

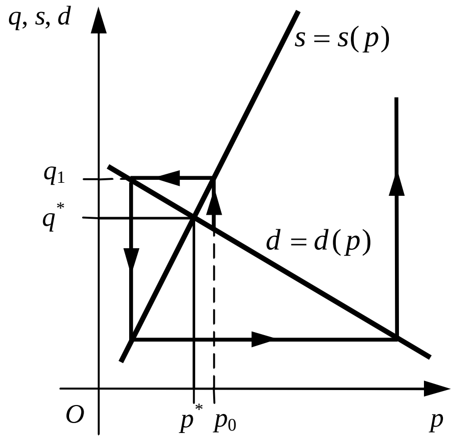

Fig. 5

The expression in square brackets is the sum of t terms of a geometric progression. We use the formula for the sum of t terms of a geometric progression: $S_t = \frac{\beta(1-r^t)}{1-r}$. If $|r|<1$, then $\lim_{t\to\infty} S_t = \frac{\beta}{1-r}$.

For this cobweb model we have $r = -\frac{b}{l}$, $\beta = 1$, whence we obtain the expression for the price p_t at any discrete moment of time t $(t = 1, 2, 3, \ldots)$:

$$p_t = \frac{c-a}{l}\frac{1-(-1)^t\left(\frac{b}{l}\right)^t}{1+\frac{b}{l}} + (-1)^t\left(\frac{b}{l}\right)^t p_0$$

or

$$p_t = \frac{c-a}{b+l} + (-1)^t \left(\frac{b}{l}\right)^t \left(p_0 - \frac{c-a}{b+l}\right),$$

or

$$p_t = p^* + (-1)^t \left(\frac{b}{l}\right)^t (p_0 - p^*). \qquad (3)$$

As can be seen from formula (3), it follows that:

If $\frac{b}{l} < 1$, i.e. the slope of the demand straight line is steeper than the slope of the supply straight line, then $\left(\frac{b}{l}\right)^t \to 0$ as $t \to \infty$, therefore, $p_t \to \frac{c-a}{b+l} = p^*$, and the equilibrium is stable.

If $\frac{b}{l} > 1$, i.e. the slope of the supply straight line is steeper than the slope of the demand straight line, then $\left(\frac{b}{l}\right)^t \to \infty$ as $t \to \infty$, process diverging (the equilibrium is unstable).

If $\frac{b}{l} = 1$, i.e. $b = l$, then the value

$$p_t = p^* + (-1)^t (p_0 - p^*) = p^* + (-1)^t \mathbb{A} \qquad (4)$$

fluctuates with the constant amplitude $\mathbb{A} = p_0 - p^*$ about the equilibrium value p^* (the indifferent equilibrium).

As we can see, the results of the mathematical analysis of this model coincide with the results of graphical analysis.

We draw graphically the dependence of p_t on t for $\frac{b}{l} = 1$ (see formula (4) and fig. 6).

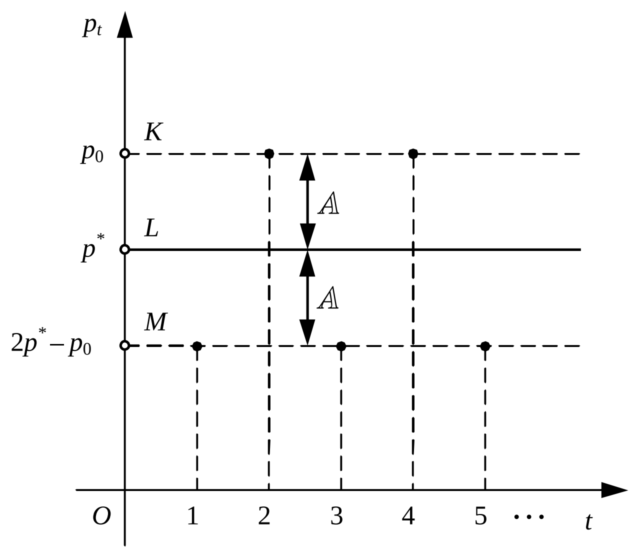

Fig. 6

As can be seen from fig. 6, it follows that:

$$KL = p_0 - p^*, \quad LM = p^* - (2p^* - p_0) = p_0 - p^*,$$

i.e. $KL = LM$. Then

$$\mathbb{A} = KL = LM = p_0 - p^*.$$

The amplitude $\mathbb{A}$ obtained corresponds to fig. 3 as well.

Conclusions. The cobweb model is explored in the article. Mathematical analysis of this model is conducted.

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